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On the Analytic Theory of Circular Functions.

BY ALEXANDER S. CHESSIN.

1.—*Preliminaries.*

§1. It has become a usage with authors of treatises on the Theory of Functions to introduce the reader to the theory of doubly periodic functions by first treating simply periodic functions. Unfortunately the similarity between simply and doubly periodic functions ceases to exist when the behavior of the function at infinity comes to be investigated. Indeed, while in the case of doubly periodic and, in particular, of elliptic functions we treat these functions in a primitive parallelogram situated in the finite portion of the plane, we have to consider, in the case of simply periodic and, in particular, of circular functions, the *behavior of such functions at infinity, when the variable is restricted to remain within one of the primitive regions or bands into which the plane may be divided*. It was the neglect of this important point in the theory of simply periodic functions that led M. Forsyth to some erroneous conclusions in his excellent treatise on the Theory of Functions.* M. Méray, in his “Leçons nouvelles sur l’Analyse infinitésimale et ses applications géométriques,”† gives considerable attention to the point in question, which leads him to a classification of simply periodic functions into *polarized* and *non-polarized* functions.‡ However, the character and role of the *polar values*§ of a circular function have not yet been clearly set forth, and it is the object of the present paper to supply this deficiency.

* See ch. X; also the review of this treatise by Mr. W. F. Osgood in the Bull. of the Amer. Math. Soc., 2d series, vol. I, no. 6.

† Vol. II, ch. VII.

‡ Op. cit. p. 270. These terms correspond to the terms *circular* and *pseudo-circular* used in this paper.

§ *Characteristic limits*, in the present paper.

§2. Let $f(z)$ be a uniform function admitting the single period ω ; then, by definition,

$$\begin{aligned} f(z + n\omega) &= f(z), \\ n &\equiv \pm 1, \pm 2, \pm 3, \dots \end{aligned} \quad (1)$$

This equation holding only for integral values of n , the period ω is *primitive*.

By means of a simple linear substitution of the variable the function $f(z)$ can be transformed into another uniform function admitting an arbitrarily assigned period ω' . In fact, if we put

$$z = \frac{\omega}{\omega'} z'; \quad f(z) = f_1(z')$$

we shall have

$$f_1(z' + \omega') = f_1(z').$$

In particular, if ω be a complex number, we may put $\omega' = |\omega|$, so that the transformed function will admit a real period.

§3. It will be shown presently that the entire plane of z may be divided into parallel bands in such a way that $f(z)$ takes all its values in each band. This proposition, the analogous of which is almost obvious in the case of doubly periodic functions, requires special considerations on account of the fact that the bands into which the plane of z is divided extend to infinity.

We will agree to say that two points z and z' are congruent to each other with respect to ω if we have

$$z \equiv z' \pmod{\omega},$$

adopting this notation from the theory of numbers.

Let us draw any arbitrary line L which does not cut itself and extends to infinity, and let a point z describe this line; then the points congruent to z will describe the system of lines

$$\dots L_{-2}, L_{-1}, L_0, L_1, L_2, \dots$$

parallel to each other and to the line L , the line L_k being described by the point $z + k\omega$, and the entire plane of z will be divided into parallel bands. Each band, i. e. the region contained between any two consecutive lines L_{k-1} and L_k , will be called a *primitive region* of the function $f(z)$. For the sake of convenience we will say that the band formed by the lines L_{k-1} , L_k is the k^{th} primitive region. To each primitive region belongs one of the two lines forming it. We will assume that to the k^{th} primitive region belongs the line L_{k-1} .

Let z be any point within a fixed primitive region, for instance the m^{th} region; all points congruent to z will lie within other primitive regions, namely the point $z + k\omega$ will lie in the $(m + k)^{\text{th}}$ region.

If z_1 be an ordinary point of $f(z)$, then all points congruent to z_1 are ordinary points of the function. In fact, in the neighborhood of the point z_1 we can develop $f(z)$ in the form

$$f(z) = G(z - z_1), \quad (2)$$

where $G(z)$ denotes, as usually, an integral function. Let $z_2 \equiv z_1 \pmod{\omega}$ and z' be any point in the neighborhood of the point z_2 , i. e. let

$$z' = z_2 + \zeta; \quad |\zeta| < \varepsilon. \quad (3)$$

Then by (1),

$$f(z') = f(z_2 + \zeta) = f(z_1 + \zeta),$$

and by (2) and (3),

$$f(z_1 + \zeta) = G(\zeta) = G(z' - z_2),$$

which proves the proposition.

If the point z_1 were an isolated singular point of $f(z)$ —and this is the only kind of singularity we need to consider for our purposes—we would have instead of formula (2) the following one:

$$f(z) = G_1(z - z_1) + G_2\left(\frac{1}{z - z_1}\right). \quad (4)$$

Again, by (1),

$$f(z') = f(z_1 + \zeta),$$

and by (4) and (3),

$$f(z_1 + \zeta) = G_1(\zeta) + G_2\left(\frac{1}{\zeta}\right) = G_1(z' - z_2) + G_2\left(\frac{1}{z' - z_2}\right),$$

i. e. the function $f(z)$ has at the points z_2 exactly the same singularity as at the point z_1 .

In a similar way it is readily shown that if z_1 be a vanishing point of the order λ for $f(z)$, all points congruent to z_1 are vanishing points of the same order for this function.

It follows from the preceding remarks that the function $f(z)$ assumes in the finite portion of a primitive region all the values which it assumes in the finite portion of the entire plane, and that $f(z)$ is completely defined in the *finite portion* of the plane if it be defined in the *finite portion* of a primitive region. But it would be wrong to conclude without further investigation that $f(z)$ is

completely defined in the *entire* plane if it be completely defined in a primitive region. It remains therefore to investigate the behavior of $f(z)$ at infinity, and to find out whether with regard to the infinity point also the function is characterized by its behavior at infinity *while z remains within a primitive region*.

§4. THEOREM I.—*The function $f(z)$ has an essentially singular point at infinity.*

This follows immediately from equation (1). In fact, if we put in it $n = \infty$ we obtain

$$f(\infty) = f(z),$$

i. e., the function assumes any arbitrarily assigned value at infinity, and the point $z = \infty$ is therefore an essentially singular point of the function.

THEOREM II.—*If the function $f(z)$ has an essentially singular point other than the point $z = \infty$, then it has an infinite number of essentially singular points.*

For if z_1 be an essentially singular point of $f(z)$ and $z_1 \neq \infty$, then all points congruent to z_1 are essentially singular points of the function (§3).

We will be concerned in this paper only with functions $f(z)$ having no other essentially singular point than the point $z = \infty$. Under this restriction $f(z)$ can have only isolated poles in the finite portion of a primitive region. In fact we know that whenever an infinite accumulation of poles takes place at a point, this point is an essentially singular point of the function. An infinite accumulation of poles may therefore take place only at infinity. This does not exclude the possibility of an infinite number of poles in a primitive region, provided there be only a finite number of them in the finite portion of such a region.

Let then $f(z)$ be a function as here defined. We can enunciate with regard to this function the following two propositions:

THEOREM III.—*If the function $f(z)$ has no pole in a primitive region, then it is an integral transcendental function, i. e.*

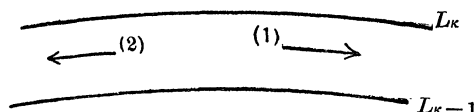
$$f(z) \equiv G(z).$$

THEOREM IV.—*If the function $f(z)$ has a finite number of poles in a primitive region or an infinite number of them, provided there be only a finite number of poles in the finite portion of the region, then $f(z)$ is the quotient of two integral functions, i. e.*

$$f(z) \equiv \frac{G_1(z)}{G_2(z)}.$$

These two theorems are an immediate consequence of Weierstrass's theory. With regard to the second theorem, it must be observed that one or the other of the functions $G_1(z)$, $G_2(z)$ may reduce to a constant, but if not constant they must be *transcendental* integral functions; because if $f(z)$ has a vanishing point or a pole in a primitive region, it will necessarily have an infinite number of vanishing points or poles in the whole plane (§3).

§5. Let $f(z)$ be again a function as defined in the preceding paragraph, and suppose that we restrict the variable to *remain within a fixed primitive region* of the function. If then we let z tend to infinity and at the same time fix the direction of motion by one of the two arrows of the figure, the path of z being



otherwise perfectly arbitrary, three possibilities arise: either $f(z)$ tends to a finite and determinate limit, or it tends to become infinite in a determinate way, or, finally, it becomes indeterminate. It will be shown later (§10) that if one of the enumerated possibilities takes place in one of the primitive regions, it will take place also in every other primitive region. It will be shown, moreover, that if one of the first two cases occurs for one of the directions indicated by the arrows, the same will be true for the other direction. Finally, it will be shown that this property is independent of the choice of the line L .

Functions $f(z)$, for which the first two of the enumerated cases take place, form a class by themselves and will be called *circular functions* in distinction from the functions for which the third of the enumerated cases takes place, and which will be called *pseudo-circular functions*.

THEOREM.—*A circular function can have only a finite number of poles in a primitive region.*

Suppose first that $\lim_{z=\infty} f(z)$ is finite and determinate when the variable remains in a primitive region and tends to infinity in the direction indicated by one of the two arrows of the figure. Let N be this limit. Then we are able to assign a *neighborhood* of the point $z = \infty$ within the primitive region such that within this neighborhood $f(z)$ differs arbitrarily little from N . Now if $f(z)$ had an infinite number of poles in a primitive region, these would form an infinite

accumulation at $z = \infty$ (§4), and it would be impossible to assign any neighborhood of $z = \infty$ in the primitive region such that in this neighborhood $f(z)$ differ arbitrarily little from N .

Suppose on the other hand that $f(z)$ tends to infinity in a determinate way when z tends to infinity in the described manner. Then we are able to assign a neighborhood of the point $z = \infty$ within the primitive region such that within this neighborhood $f(z)$ have arbitrarily great but *finite* values. This again would be impossible if $f(z)$ had an infinite number of poles in a primitive region.

REMARK.—It will prove convenient to combine the two cases, namely, the case when $\lim_{z=\infty} f(z)$ is finite and determinate, and the case when $f(z)$ tends to infinity in a determinate way, under the same category of *functions having a determinate limit* when z tends to infinity in the manner described above. We will therefore agree that, unless explicitly expressed (*finite* and determinate), $\lim_{z=\infty} f(z)$ may be infinite when we say that it has a determinate limit. It will, however, be remembered that this is only a convenient mode of expressing that $f(z)$ may *tend to infinity in a determinate way*.

2.—*Behavior of Circular Functions at Infinity.*

§6. Let us consider a circular function having a real period Ω . We will choose for the line L the axis of y , and we will assume that $\lim_{z=\infty} f(z)$ has a determinate value when $z = x + iy$ tends to infinity while remaining in the first primitive region, the direction of motion being in this case indicated by the sign of y . We will denote the limits of $f(z)$ by f_1 and f_2 according as $y > 0$ or $y < 0$. These values may be infinite.* Let us next draw any arbitrary line Λ extending to infinity. The angle formed by the direction of the motion of a point z on this line and the positive axis of y , is a function of z , which we will denote by $\lambda(z)$. As the point z moving on the line Λ tends to infinity, the function $\lambda(z)$ may tend to a determinate limit, which we will denote by $\lambda(\infty)$, or it may become indeterminate.

THEOREM I.—If $\lambda(z)$ tends to a determinate limit $\lambda(\infty) \neq \frac{\pi}{2}$, when z tends to

* See Remark, §5.

infinity along the line Λ , then $\lim_{z=\infty} f(z) = f_1$ or f_2 , according as $\lambda(\infty) < \frac{\pi}{2}$ or $\lambda(\infty) > \frac{\pi}{2}$.

Let $z = x + iy$ be a point on the line Λ and $z_0 = x_0 + iy_0$ its congruent in the first primitive region. Then

$$\begin{aligned} x &\equiv x_0 \pmod{\Omega}; \quad y = y_0, \\ f(z) &= f(x_0 + iy). \end{aligned}$$

Let z tend to infinity along the line Λ . Then x_0 will remain within the limits

$$0 < x_0 < \Omega,$$

and $x_0 + iy$ will remain within the first primitive region. Hence

$$\lim_{z=\infty} f(z) = \lim_{y=\pm\infty} f(x_0 + iy) = f_1 \text{ or } f_2,$$

according as $y = +\infty$ or $-\infty$, i. e. according as $\lambda(\infty) < \frac{\pi}{2}$ or $> \frac{\pi}{2}$, which proves the proposition.

REMARK.—*Theorem I still holds for $\lambda(\infty) = \frac{\pi}{2}$ provided x and y both tend to infinity with z . The value of $\lim_{z=\infty} f(z)$ in this case will be f_1 or f_2 according as $\lambda(z)$ remains from a certain place on $< \frac{\pi}{2}$ or $> \frac{\pi}{2}$ as z tends to infinity along the line Λ .*

THEOREM II.—*If $\lambda(z)$ tends to the determinate limit $\lambda(\infty) = \frac{\pi}{2}$, but y does not become infinite with z , then $\lim_{z=\infty} f(z)$ becomes indeterminate as z tends to infinity along Λ .*

In this case the line Λ has an asymptote parallel to the axis of x .* Let a be its distance from this axis. Then it is clear that as z tends to infinity along the line Λ , x_0 will again remain within the limits

$$0 \leq x_0 < \Omega,$$

while y will tend (or be equal) to the value a , so that

$$\lim_{z=\infty} f(z) = f(x_0 + ia),$$

* Or Λ may simply become parallel to the axis of x .

where x_0 may have any value between 0 and Ω . Therefore $\lim_{z=\infty} f(z)$ may have any value among those which $f(z)$ assumes on the line $y=a$ within the first primitive region, i. e. $\lim_{z=\infty} f(z)$ is indeterminate, which proves the theorem.

We can of course imagine that z tends to infinity not along a continuous line, but by jumps from point to point. In this case it is no more possible to speak of an angle between the direction of the motion of the point z and the positive axis of y . But it is always possible to draw a line Λ such that the moving point will remain on this line, and as the preceding theorems are independent of the way in which the point z may move on the line Λ , it is clear that they still remain true. The function $\lambda(z)$ will depend on the manner in which the line Λ is drawn through the points passed by the moving point, but $\lambda(\infty)$ will be the same (if determinate) for all such lines Λ , because they will all be tangent to each other at the point $z = \infty$.

Heretofore we have assumed that $\lambda(\infty)$ had a determinate value. It remains to examine such paths Λ for which $\lambda(\infty)$ is indeterminate. It is clear that all such paths may be divided into two classes. The paths of the first class are such that as z tends to infinity along one of them $\lambda(z)$ may assume any value between 0 and π *except the value* $\frac{\pi}{2}$. In this case $\lim_{z=\infty} f(z)$ will oscillate indefinitely between the values f_1 and f_2 . The paths of the second class are characterized by the fact that as z tends to infinity along one of them $\lambda(z)$ *may among other values assume the value* $\frac{\pi}{2}$.

THEOREM III.—*Whatever be the path along which z tends to infinity, the value of $\lim_{z=\infty} f(z)$ will be either f_1 or f_2 or one of the values which $f(z)$ assumes in the finite portion of the first primitive region.*

In fact if $\lambda(\infty)$ has a determinate value, if this value be $\neq \frac{\pi}{2}$ we have seen that $\lim_{z=\infty} f(z) = f_1$ or f_2 . If $\lambda(\infty) = \frac{\pi}{2}$, the value of $\lim_{z=\infty} f(z)$ is any one of the values of $f(z)$ on a line parallel to the axis of x within the first primitive region, i. e. one of the values which $f(z)$ assumes in the finite portion of the first primitive region.

If $\lambda(\infty)$ is indeterminate but $\neq \frac{\pi}{2}$, we have seen that $\lim_{z=\infty} f(z)$ can have only the values f_1 and f_2 , and if $\lambda(\infty)$ is entirely indeterminate, $\lim_{z=\infty} f(z)$ cannot only have the values f_1 and f_2 but also any value among those which $f(z)$ assumes on a line parallel to the axis of x within the first primitive region, i. e. $\lim_{z=\infty} f(z)$ can have besides f_1 and f_2 any value of $f(z)$ in the finite portion of the first primitive region. Q. E. D.

§7. We have assumed at the beginning of the preceding paragraph that the circular function $f(z)$ with the real period Ω has a determinate limit when z tends to infinity in any manner within the *first* primitive region, provided the sign of y in $z = x + iy$ be ultimately the same,* and we have called this limit f_1 or f_2 according as the sign of y is positive or negative. Let us now see how the function behaves at infinity in any other primitive region.

Suppose z be restricted to remain within the k^{th} primitive region, and let z tend to infinity in such a way that y have ultimately a fixed sign. Then it is clear that we can apply Theorem I of the preceding paragraph. In fact in this case $\lambda(\infty) = 0$ or π according as y becomes ultimately positive or negative. In the first case $\lim_{z=\infty} f(z) = f_1$, in the second $\lim_{z=\infty} f(z) = f_2$. Hence this

THEOREM I.—*If a function $f(z)$ with a real period tends to a determinate limit when z tends to infinity while remaining in a fixed primitive region in such a way that y have ultimately a determinate sign, then the function will tend to the same limit when z tends to infinity while remaining in any other primitive region and the sign of y being ultimately the same as in the first case.*

Now that we have seen that $\lim_{z=\infty} f(z)$ has the same value in every one of the primitive regions, it remains to find whether this property is independent of the manner in which the plane of z has been divided into primitive regions. Let therefore L be any line which does not cut itself and extends to infinity, and which has no asymptote parallel to the axis of x or which does not ultimately coincide with a line parallel to this axis. If z be restricted to remain within a fixed primi-

* This condition which here replaces the necessity of choosing one of the two arrows on the fig. of §5 is necessary, for otherwise $f(z)$ would oscillate indefinitely between the values f_1 and f_2 as z tends to infinity.

tive region in this new division of the plane, it is clear that we can again apply Theorem I of the preceding §, and we shall have

$$\lim_{z=\infty} f(z) = f_1 \text{ or } f_2$$

according as y becomes ultimately positive or negative. Hence this result:

The preceding theorem remains true whatever be the manner in which the plane has been divided into primitive regions.

It will be convenient for the future developments to fix a *positive direction of the line L* . In the present case (i. e. when the period of $f(z)$ is real) we will agree to take as positive that direction of the line L which at infinity forms an acute angle with the positive axis of y . We will also agree to say that z *tends to infinity in the positive* (resp. negative) *direction, if the direction of its motion forms ultimately an acute* (resp. obtuse) *angle with the positive direction of the line L* . It is clear that if z remains within a fixed primitive region, the limit of this angle is 0 (resp. π).

With these new definitions we can enunciate Theorem I in a more general form as follows:

THEOREM II.—*If a function $f(z)$ with a real period tends to a determinate limit when z tends to infinity in a positive (resp. negative) direction while remaining in a fixed primitive region for a fixed division of the plane, then the function $f(z)$ will tend to the same limit within every one of the primitive regions, as well for the given as for any other division of the plane, provided z tend to infinity always in the positive (resp. negative) direction.*

§8. The preceding discussion shows that with every circular function having a real* period are connected two fixed numbers f_1 and f_2 † which are inasmuch characteristic for the function that they are independent of the manner in which the plane is divided into primitive regions, and that they are obtained as the limiting values of the function when the variable tends to infinity along any path having a determinate direction (as defined above), this limiting value being f_1 or f_2 according as the variable tends to infinity in the positive or negative

* We shall see later that this restriction of the period is not necessary.

† These numbers may be equal to each other. One or both of them may also be infinite. (See Remark at the end of §5.)

direction. We will therefore call these numbers f_1 and f_2 *the characteristic limits at infinity*, or shorter, *the characteristic limits of the circular function $f(z)$* .

From Theorem III, §6, and Theorem II, §7, follows an important result, namely: *The function $f(z)$ whose period is real, assumes in a primitive region every assignable value.*

In fact, by Theorem III, §6, whatever be the path of z , the value of $\lim_{z=\infty} f(z)$ will be either f_1 or f_2 or any one of the values which $f(z)$ assumes in the finite portion of a fixed primitive region for a fixed division of the plane. By Theorem II, §7, the values f_1 and f_2 are the limits of $f(z)$ for $z = \infty$ also in any other primitive region and independently of the manner in which the plane is divided into parallel bands. We know on the other hand that all the values assumed by $f(z)$ in the finite portion of the entire plane are also assumed by the function in the finite portion of any one of its primitive regions, independently of the division of the plane (§3). Hence all the values of $\lim_{z=\infty} f(z)$ are those which $f(z)$ assumes in a primitive region, the numbers f_1 and f_2 , i. e. the values of $f(z)$ at infinity in the primitive region included. But we know that at the point $z = \infty$, which is an essentially singular point of $f(z)$, the function assumes every assignable value; hence $f(z)$ assumes every assignable value in a primitive region.

Q. E. D.

If we recall the manner in which the several values of $\lim_{z=\infty} f(z)$ are obtained, we will notice that all the values except possibly f_1 and f_2 can be obtained by making z tend to infinity along lines having asymptotes parallel to the axis of x , and the distances of these asymptotes from the axis of x assuming every possible negative or positive value. The simplest manner of drawing these lines is to take them parallel to the axis of x . We can therefore say that

Every assignable value except possibly the characteristic limits of a circular function having a real period, can be obtained for $\lim_{z=\infty} f(z)$ by making z tend to infinity along a line parallel to the axis of x .

We shall see later (§10), that in the general case we only need to substitute for the axis of x the line forming with it the angle $\arg(\omega)$, ω being the period of the function.

§9. When we defined the function $f(z)$ in §6 we assumed that $\lim_{z=\infty} f(z)$ had a determinate value for either of the two directions in which z may tend to

infinity in a primitive region. The question naturally presents itself: is it necessary to assume that $\lim_{z=\infty} f(z)$ is determinate for either direction, or does it suffice to obtain a determinate limit for only one direction to assure the determinateness of the limit for the other?

THEOREM.—*If a function $f(z)$ having a real period tends to a determinate limit when z tends to infinity in the positive direction while remaining in a primitive region, then the function will also tend to a determinate limit when z tends to infinity in the negative direction.*

In fact, suppose that the function tends to a determinate limit when z tends to infinity along a given fixed path Λ in the positive direction. To this path we can correlate another Λ' symmetrical with respect to the axis of x . Two correlated points on Λ and Λ' are then $x + iy$ and $x - iy$. Now, we can always give to the function $f(z)$ the form

$$f(z) = U(z) + iV(z), \quad (5)$$

where $U(z)$ and $V(z)$ are *real* functions of the variable z as long as z is *real*. At the same time we may put

$$\begin{cases} U(z) = U_1(x, y) + iU_2(x, y), \\ V(z) = V_1(x, y) + iV_2(x, y), \end{cases} \quad (6)$$

so that if

$$f(z) = [U_1(x, y) - V_2(x, y)] + i[U_2(x, y) + V_1(x, y)] \quad (7)$$

gives the value of the function for a point (x, y) of Λ , the value of the function for the correlated point on Λ' will be

$$f(z) = [U_1(x, y) + V_2(x, y)] + i[-U_2(x, y) + V_1(x, y)]. \quad (8)$$

A simple glance at the expressions (7) and (8) shows that if one of them tends to a determinate limit when z tends to infinity, the other will also tend to a determinate limit.

Now if along Λ z tends to infinity in the positive direction, it is clear that along Λ' it tends to infinity in the negative direction. But as $\lim_{z=\infty} f(z)$ is determinate when z tends to infinity in the positive direction while remaining in a primitive region, $\lim_{z=\infty} f(z)$ will have a determinate value whatever be the path

Λ along which z tends to infinity* in the positive direction (Theor. II, §7). Hence $\lim_{z=\infty} f(z)$ will have a determinate value whatever be the path Λ' along which z tends to infinity* in the negative direction. Q. E. D.

REMARK.—It is obvious that the functions $U(z)$ and $V(z)$ are simply periodic and have the same period as $f(z)$. Moreover $\lim_{z=\infty} U(z)$ and $\lim_{z=\infty} V(z)$ have a determinate value when z tends to infinity in the manner described above. Hence $U(z)$ and $V(z)$ are *circular functions* like $f(z)$.

COROLLARY I.—*We can determine the characteristic limits of a circular function $f(z) = U(z) + iV(z)$ having a real period if we know one of the characteristic limits for each of the functions $U(z)$ and $V(z)$, provided these be not both infinite.*

In fact suppose, to fix the ideas, that we know the characteristic limit of $U(z)$ when z tends to infinity in the positive direction, and let it be $u' + iu''$; also the characteristic limit of $V(z)$ for instance when z tends to infinity in the negative direction, and let it be $v' + iv''$. Then, by formulas (6),

$$\begin{aligned} u' &= \lim_{z=\infty} U_1; & u'' &= \lim_{z=\infty} U_2, \\ v' &= \lim_{z=\infty} V_1; & v'' &= -\lim_{z=\infty} V_2, \end{aligned}$$

and therefore by (7) and (8),

$$\left. \begin{aligned} f_1 &= u' + v' + i(u'' + v''), \\ f_2 &= u' - v' + i(-u'' + v''). \end{aligned} \right\} \quad (9)$$

If only one of the numbers $u' + iu''$ or $v' + iv''$ be infinite, the corollary still holds. In fact then $f_1 = f_2 = \infty$, and the proposition is thus proved.

REMARK.—If both $u' + iu''$ and $v' + iv''$ were infinite, it would be necessary to go back to the expressions (7) and (8) and then make z tend to infinity in order to obtain the limits f_1 and f_2 .

COROLLARY II.—*If the characteristic limits f_1 and f_2 of a circular function having a real period are finite, then the condition necessary and sufficient in order that $f_1 = f_2$ is that the characteristic limits of the functions $U(z)$ and $V(z)$ be real.*

This follows immediately from formulas (9), which give the condition $v'' = u'' = 0$.

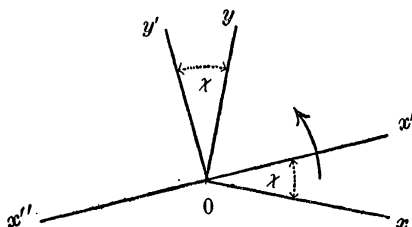
* Always excluding such lines as tend to become parallel to the axis of x at infinity.

The discussion of this paragraph shows that the propositions advanced when defining circular functions in §5 are true at least in the case of a real period. It will be shown presently that they are always true.

§10. Heretofore it has been assumed that the period of the circular function was real (§6). But we can at once extend our results to the general case when the period ω of $f(z)$ is any complex number if we make use of the linear substitution mentioned in §2. In fact, we put again

$$\Omega = |\omega|; \quad z = \frac{\omega}{\Omega} z'; \quad f(z) = F(z'),$$

and then $F(z')$ is a circular function admitting the real period Ω . We may therefore apply the results of §§6–9 to the function $F(z')$. It will be noticed that the positive axis of x' forms the angle $\chi \equiv \arg(\omega)$ with the positive axis of x . Positive angles will be counted in the direction indicated by the arrow on the figure. The line $x'x''$ will be called *the line of periodicity of the function*



$f(z)$, and the direction $0x'$ the *positive direction of the line of periodicity*. We will also agree to call the direction $0y'$ the *normal to the line of periodicity*.

With these definitions it becomes clear that we only need to substitute the line of periodicity for the axis of x in the results of §§6–9 to make them valid in the general case, i. e. when the period of $f(z)$ is any complex number.

The function $\lambda(z)$ used in §§6 and 7 will now be the angle between the direction of the motion of z when it tends to infinity along a line Λ , and the normal to the line of periodicity. The enunciation of the three theorems of §6 need not be changed.

Coming to §7 it is necessary first to agree as to the terms “positive direction of L ” and “ z tends to infinity in the positive direction.” The positive direction of L will be that which at infinity forms an acute angle with the normal to the line of periodicity; and we will say that z tends to infinity in the positive (resp.

negative) direction if the direction of its motion ultimately forms an acute (resp. obtuse) angle with the positive direction of the line L . It will be assumed that the line L has no asymptotes parallel to the line of periodicity (or that it does not ultimately coincide with a line parallel to it).

With these more general definitions we can at once extend Theorem II §7 to functions $f(z)$ with a complex period ω . As to Theorem I, it is only another form of the same proposition. The results of §8, namely, the *existence of two characteristic limits*, and the proposition that *a circular function assumes every assignable value in a primitive region*, are immediately extended to any circular function. As to the last proposition of §8 it will now be formulated as follows:

Every assignable value except possibly the characteristic limits of a circular function $f(z)$ can be obtained for $\lim_{z=\infty} f(z)$ by making z tend to infinity along a line parallel to the line of periodicity.

The Theorem of §9 obviously holds for any circular function; to formulate it for the general case again we only need to substitute the line of periodicity for the axis of x . As to the corollaries to this theorem, they evidently hold only for circular functions with a real period. However they may still be used with advantage. In fact the characteristic limits of the functions $f(z)$ and $F(z')$ being identical, we may by applying these corollaries to the function $F(z')$, either determine the characteristic limits of $f(z)$ (Cor. I) or decide whether these limits are equal (Cor. II).

It follows from the preceding discussion that the propositions advanced when defining circular functions in §5 are true. This definition may therefore be formulated as follows:

DEFINITION.—*A circular function is a uniform simply periodic function with no other essentially singular point than the point $z = \infty$ and such that $\lim_{z=\infty} f(z)$ has a determinate value when z tends to infinity in a fixed direction while remaining in a primitive region.*

A function so defined tends to a determinate limit when z tends to infinity along any fixed path in a fixed direction; moreover it possesses two characteristic limits which may be equal.

It also follows from the preceding discussion that a pseudo-circular function cannot tend to a determinate limit when z tends to infinity while remaining within a primitive region, *whatever be the division of the plane*, i. e. whatever be

the line L . But a pseudo-circular function may tend to a determinate limit when z tends to infinity along some particular paths within a primitive region. For example, the function $e^{\sin z}$ tends to zero when z tends to infinity along parallels to the axis of y at the distances $(4n + 3) \frac{\pi}{2}$ where $n \equiv 0, \pm 1, \pm 2, \dots$

As another example of a pseudo-circular function may be mentioned Hermite's function $Z(z)$; its derivative is an elliptic function. The function $Z(z)$ tends to no determinate value when z tends to infinity remaining in a primitive region, whatever be the path chosen.

It would be interesting to investigate more closely the character of this distinction of pseudo-circular functions, but this paper is restricted to the study of circular functions alone.

§11. THEOREM.—*A circular function which is finite everywhere in a primitive region is a constant.*

In fact suppose that the function $f(z)$ is finite in the finite portion of a primitive region and that its characteristic limits f_1 and f_2 are also finite. All the values of $f(z)$ at infinity except perhaps the values f_1 and f_2 can be obtained by making z tend to infinity along lines parallel to the line of periodicity (§10). But $f(z)$ being finite everywhere in a primitive region, the values of $f(z)$ at infinity so obtained are necessarily also finite; hence all the values of $f(z)$ at infinity are finite, i. e. the function is holomorphic in the entire plane of z , and therefore it reduces to a constant.

COROLLARY I.—*A circular function becomes infinite at least once within a primitive region.*

The function may have one or more poles in a primitive region or one or both of its characteristic limits may be infinite. If $f(z)$ has no pole in a primitive region, then at least one of its characteristic limits is infinite. If both its characteristic limits are finite, $f(z)$ has at least one pole in a primitive region.

COROLLARY II.—*A circular function $f(z)$ assumes every assignable value at least once in a primitive region.*

This follows at once from the preceding Corollary if we consider the circular function

$$\frac{1}{f(z) - A},$$

where A is any arbitrarily assigned number.

§12. *Picard's theorem.*

If we recollect how the different values of $f(z)$ at infinity may be obtained, we see that there will be an infinite number of points in the neighborhood of the point $z = \infty$, at which the function assumes any arbitrarily assigned value except perhaps the values of the characteristic limits: f_1 and f_2 . In fact all other values may be obtained by making z tend to infinity along lines parallel to the line of periodicity (§10), and each one of these values will repeat itself an infinite number of times as z tends to infinity. We thus obtain Picard's theorem* for the particular case of circular functions in the following form:

In the neighborhood of the essentially singular point of a circular function there is an infinite number of points at which the function assumes any arbitrarily assigned value except possibly one or two values.

We may add that—

these exceptional values are the characteristic limits of the circular function, and that there will be one or two exceptional values according as the characteristic limits are or are not equal.

We may furthermore add that—

all values of $\lim_{z=\infty} f(z)$ other than f_1 and f_2 may be obtained by making z tend to infinity along lines parallel to the line of periodicity, and that the values f_1 and f_2 will or will not be exceptional according as the function does not or does assume these values in the finite portion of a primitive region; and finally, that either the function assumes the values f_1 and f_2 at an infinite number of points in the neighborhood of the point $z = \infty$, or it assumes them only at the point $z = \infty$ itself.

§13. We are able now to answer the question formulated at the end of §3, namely, in the affirmative sense: with regard to the point $z = \infty$ as with regard to any other point a circular function is characterized by its behavior in a primitive region, and we can enunciate the following fundamental

THEOREM.—*A circular function is defined in the entire plane if it is defined in a primitive region.*

This theorem is independent of the manner in which the plane is divided into parallel bands.

We may therefore restrict our investigation to the study of circular functions in a primitive region.

* "Mémoire sur les Fonctions entières," An. de l'École Normale, 1880.

3.—*Study of Circular Functions within a Primitive Region.*

§14. THEOREM I.—*If two circular functions $f(z)$ and $\phi(z)$ admitting the same period ω and whose characteristic limits at infinity are finite and $\neq 0$ have the same vanishing points and poles with the same respective orders of multiplicity in a primitive region, then*

$$f(z) = C\phi(z).$$

In fact the function $\frac{f(z)}{\phi(z)}$ is a circular function which becomes infinite nowhere in a primitive region, and therefore reduces to a constant (§11).

THEOREM II (generalization of the preceding theorem).—*If two circular functions $f(z)$ and $\phi(z)$ admitting the same period ω have the same vanishing points and poles with the same respective orders of multiplicity in a primitive region, and if at the same time the characteristic limits at infinity of the circular function $\frac{f(z)}{\phi(z)}$ are finite and $\neq 0$, then $f(z) = C\phi(z)$.*

In fact $\frac{f(z)}{\phi(z)}$ is again finite everywhere in a primitive region and therefore reduces to a constant.

THEOREM III.—*If two circular functions admitting the same period and whose characteristic limits at infinity are finite, have the same poles with the same respective orders of multiplicity in a primitive region; moreover, if β_i being any one of these poles and μ_i its order of multiplicity, the coefficients of the different powers of $\frac{1}{z - \beta_i}$ in the developments of the two functions in the neighborhood of the point β_i be respectively the same, then*

$$f(z) = \phi(z) + C.$$

In fact the circular function $f(z) - \phi(z)$ is finite throughout a primitive region, and therefore reduces to a constant.

This theorem may also be generalized as follows :

THEOREM IV.—*If two circular functions admitting the same period have the same poles with the same respective orders of multiplicity in a primitive region; moreover, if β_i being any one of these poles and μ_i its order of multiplicity, the coefficients of the different powers of $\frac{1}{z - \beta_i}$ in the developments of the two functions in the*

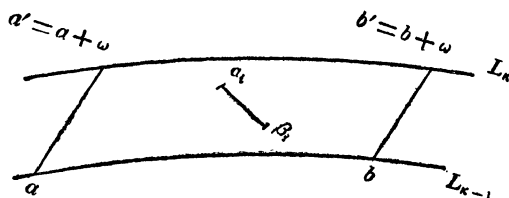
neighborhood of the point β_i be respectively the same; and if at the same time the characteristic limits of the circular function $f(z) - \phi(z)$ be finite, then

$$f(z) = \phi(z) + C.$$

The demonstration is again based on the fact that the circular function $f(z) - \phi(z)$ is finite throughout a primitive region.

§15. THEOREM.—A circular function $f(z)$ takes every assignable value except possibly f_1 and f_2 the same number of times in a primitive region. In particular it vanishes as many times as it becomes infinite in a primitive region, except possibly when f_1 or f_2 are 0 or ∞ .

We will assume at first that the characteristic limits of the function are finite and $\neq 0$. The line L being drawn arbitrarily, we can assume that it passes through no pole or vanishing point of the function. Then none of the lines L_k will pass through a pole or vanishing point of the function.



On the other hand, as f_1 and f_2 are finite and $\neq 0$, we may choose two points a and b on the line L_{k-1} in such a manner that all those points of the x^{th} primitive region which are poles (β_i) or vanishing points (α_i) of $f(z)$ will lie inside of the curvilinear parallelogram ($abb'a'$).

Consider the integral

$$\frac{1}{2\pi i} \int \text{d} \lg f(z) \quad (10)$$

taken along the boundary of this parallelogram. The path of integration may be broken up into four parts: 1) from a to b ; 2) from b to b' ; 3) from b' to a' ; 4) from a' to a . It is obvious that the first and the third of these integrals cancel each other, since the function has the same value at congruent points. As to the remaining two integrals, each one vanishes separately. In fact

$$\int_b^{b+\omega} \text{d} \lg f(z) = \lg f(b + \omega) - \lg f(b) = 0,$$

because we know that this integral is single-valued as long as the path of integration does not cross any of the lines $\alpha_i\beta_i$ connecting the poles and vanishing points of $f(z)$. For a similar reason

$$\int_{a+\omega}^a d\lg f(z) = 0,$$

and therefore the integral (10) vanishes. But we know that this integral is equal to the difference between the number of times n that $f(z)$ vanishes and the number of times m that it becomes infinite inside of the curvilinear parallelogram $(abb'a')$. Hence $m = n$, and as $f(z)$ neither vanishes nor becomes infinite in the remaining portion of the primitive region, the theorem is partly proved. To complete the proof, consider the circular function

$$\phi(z) = f(z) - N,$$

where N is any arbitrarily assigned number other than f_1 and f_2 . Then we can apply to the function $\phi(z)$ the proposition just proved, namely, $\phi(z)$ vanishes as many times as it becomes infinite in a primitive region. But $\phi(z)$ and $f(z)$ become infinite the same number of times (n) in a primitive region. Hence $f(z)$ assumes the value N also exactly n times in a primitive region. Q. E. D.

Suppose now that one or both of the characteristic limits of $f(z)$ may be 0 or ∞ . Consider in this case the circular function

$$F(z) \equiv A + \frac{1}{f(z) + B},$$

where A and B are any fixed numbers $\neq 0$ satisfying the condition $AB + 1 \neq 0$. Then the characteristic limits F_1 and F_2 of the function $F(z)$ are finite and $\neq 0$. It has been just proved that $F(z)$ assumes any arbitrarily assigned value except possibly F_1 and F_2 the same number of times in a primitive region. Hence the circular function $f(z)$ assumes any arbitrarily assigned value, except possibly f_1 and f_2 the same number of times in a primitive region. Q. E. D.

The number of times that a circular function $f(z)$ assumes in a primitive region any arbitrarily assigned value except possibly f_1 and f_2 is therefore a constant. This constant will be called *the order of the circular function*.

§16. Let $\theta(z)$ be a circular function of the first order admitting the period ω , and let this function be finite and $\neq 0$ in the finite portion of a primitive region. Then one of its characteristic limits is ∞ and the other is 0. The existence of

such functions can be established *a priori*. In fact such is, for example, the function

$$1 + \frac{z}{1} + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots, \quad (11)$$

which is denoted by the symbol e^z and whose period is $2\pi i$.* The existence of this function will be proved also *a posteriori* in §29.

Let $f(z)$ be any other circular function of the first order admitting the period ω' . We can always choose two numbers A and B such that the characteristic limits of the circular function of the first order

$$F(z) = A + \frac{1}{f(z) + B} \quad (12)$$

be finite and $\neq 0$. In fact, we only need to satisfy the conditions

$$B \neq \begin{Bmatrix} -f_1 \\ -f_2 \end{Bmatrix}; \quad \frac{AB + 1}{A} \neq \begin{Bmatrix} -f_1 \\ -f_2 \end{Bmatrix}.$$

Then the function $F(z)$ has a pole and a vanishing point both of the first order in a primitive region. Let α be the vanishing point and β the pole in the same primitive region. We can by means of the function $\theta(z)$ construct a circular function of the first order having the period ω' , whose characteristic limits are finite and whose vanishing point and pole in a primitive region are respectively α and β . In fact, such is the function

$$\Theta(z) \equiv \frac{\theta\left(\frac{\omega}{\omega'} z\right) - \theta\left(\frac{\omega}{\omega'} \alpha\right)}{\theta\left(\frac{\omega}{\omega'} z\right) - \theta\left(\frac{\omega}{\omega'} \beta\right)}. \quad (13)$$

Hence, by Theorem I, §14,

$$F(z) = C\Theta(z),$$

and therefore

$$f(z) = \frac{m\theta\left(\frac{\omega}{\omega'} z\right) + n}{p\theta\left(\frac{\omega}{\omega'} z\right) + q}, \quad (14)$$

* That all the properties of the exponential function can be derived directly from its definition by the infinite series (11) is shown in most of the modern text-books on the elements of the Theory of Functions. See, for example, Méray, *op. cit.* II, p. 207.

in which we have put

$$\begin{aligned} m &= 1 + AB - BC, \\ n &= BC\theta\left(\frac{\omega}{\omega'}\alpha\right) - (1 + AB)\theta\left(\frac{\omega}{\omega'}\beta\right), \\ p &= C - A, \\ q &= A\theta\left(\frac{\omega}{\omega'}\beta\right) - C\theta\left(\frac{\omega}{\omega'}\alpha\right). \end{aligned}$$

Formula (14) expresses the following

THEOREM I.—*Every circular function of the first order can be expressed as a linear fractional function of a given circular function of the first order, which is finite and $\neq 0$ in the finite portion of a primitive region.*

More generally we have the

THEOREM II.—*Every circular function of the first order can be expressed as a linear fractional function of any other circular function of the first order.*

In fact, if $\phi(z)$ were a given circular function of the first order with the period ω'' , we would have by formula (14)

$$\phi\left(\frac{\omega''}{\omega'}z\right) = \frac{m'\theta\left(\frac{\omega}{\omega'}z\right) + n'}{p'\theta\left(\frac{\omega}{\omega'}z\right) + q'},$$

and the elimination of $\theta\left(\frac{\omega}{\omega'}z\right)$ between the last equation and (14) gives an expression of the form

$$f(z) = \frac{m''\phi\left(\frac{\omega''}{\omega'}z\right) + n''}{p''\phi\left(\frac{\omega''}{\omega'}z\right) + q''},$$

which proves the theorem.

If we put $\theta(z) = e^z$ formula (14) takes the particular form

$$f(z) = \frac{me^{\frac{2\pi i}{\omega'}z} + n}{pe^{\frac{2\pi i}{\omega'}z} + q}. \quad (15)$$

COROLLARY.—*All circular functions of the first order which remain finite and $\neq 0$ in the finite portion of a primitive region have the form*

$$Ce^{kz}.$$

In fact, let $f(z)$ be a circular function of the first order which remains finite in the finite portion of a primitive region. Then its characteristic limits are 0 and ∞ . Formula (15) shows that in this case either $m = q = 0$ or $n = p = 0$, so that $f(z)$ has one of the two forms

$$\frac{m}{q} e^{\frac{2\pi i}{\omega'} z} \quad \text{or} \quad \frac{n}{p} e^{-\frac{2\pi i}{\omega'} z},$$

which proves the proposition.

§17. Formula (14) shows that

The characteristic limits at infinity of a circular function of the first order are different from any of the values of the function in the finite portion of a primitive region. It also shows that *the two characteristic limits of such a function are different.*

Picard's theorem (§12) in this case reads as follows: in the neighborhood of the essentially singular point of a circular function of the first order $f(z)$ there is an infinite number of points at which the function assumes any arbitrarily assigned value except the two values f_1 and f_2 , which are different and which the function assumes only at the point $z = \infty$ itself.

§18. We will use the exponential function to derive a criterion for the manner in which a circular function tends to its characteristic limits. We will denote the function $e^{\frac{2\pi i}{\omega} z}$ by the symbol u_ω , and we will say that the positive direction of the line L is that for which $\lim_{z=\infty} u_\omega = 0$. This is in accordance with the former definition of the positive direction of the line L (§10).

Let now $f(z)$ be a circular function with the period ω , whose characteristic limit in the *negative* direction is infinite. If a positive integer n can be found such that

$$\lim_{z=\infty} \left[\frac{f(z)}{u_\omega^n} \right] = N \begin{cases} \neq 0 \\ \neq \infty \end{cases}, \quad (16)$$

when z tends to infinity in the negative direction, we will say that *the characteristic limit of the function $f(z)$ is an exponential infinity of the n^{th} order.* If the characteristic limit of $f(z)$ in the *positive* direction is infinite and a positive integer n can be found such that

$$\lim_{z=\infty} [u_\omega^n f(z)] = N \begin{cases} \neq 0 \\ \neq \infty \end{cases}, \quad (16)$$

when z tends to infinity in the positive direction, we will say again that *the characteristic limit of $f(z)$ is an exponential infinity of the n^{th} order*.

If the characteristic limit of $f(z)$ is zero, we will say that it is *an exponential zero of the n^{th} order* if a positive integer n can be found such that

$$\lim_{z=\infty} [w_\omega^n f(z)] = N \left\{ \begin{array}{l} \neq 0 \\ \neq \infty \end{array} \right., \quad (17)$$

or that

$$\lim_{z=\infty} \left[\frac{f(z)}{w_\omega^n} \right] = N \left\{ \begin{array}{l} \neq 0 \\ \neq \infty \end{array} \right., \quad (17)'$$

according as we deal with a characteristic limit of $f(z)$ in the negative or in the positive direction.

More generally, if a characteristic limit of $f(z)$ is the number A , we will say that *the function $f(z)$ assumes at this limit the value A exponentially n times if a positive integer n can be found such that*

$$\lim_{z=\infty} [w_\omega^n (f(z) - A)] = N \left\{ \begin{array}{l} \neq 0 \\ \neq \infty \end{array} \right., \quad (18)$$

or that

$$\lim_{z=\infty} \left[\frac{f(z) - A}{w_\omega^n} \right] = N \left\{ \begin{array}{l} \neq 0 \\ \neq \infty \end{array} \right., \quad (18)'$$

according as we deal with a characteristic limit of $f(z)$ in the negative or in the positive direction.

These definitions are in perfect harmony with the ordinary definitions of vanishing points and poles. In fact, if the function $f(z)$ has a vanishing point α of the order λ or a pole β of the order μ , it is clear that in the first case

$$\lim_{z=\alpha} \frac{f(z)}{[e^z - e^\alpha]^\lambda} = N \left\{ \begin{array}{l} \neq 0 \\ \neq \infty \end{array} \right.,$$

and in the second case

$$\lim_{z=\beta} [(e^z - e^\beta)^\mu f(z)] = N \left\{ \begin{array}{l} \neq 0 \\ \neq \infty \end{array} \right..$$

These ordinary definitions fail at the point $z = \infty$. But they may be extended to this point, provided the variable be restricted to approach it in a certain manner only, namely, along any path which does not tend to become parallel to the line of periodicity at infinity.

This restriction being imposed only on the point $z = \infty$, it is clear that we can omit the word "exponentially" when we speak of $f(z)$ assuming a certain

value at one of its characteristic limits. Indeed, the specification of the place at which $f(z)$ assumes this value, namely, the words "at the characteristic limits," implies that the variable is restricted to remain within a primitive region. Likewise we may omit the word "exponential" when speaking of the zeros and the infinities of the function $f(z)$ if we specify how many of them are situated at the characteristic limits. For instance, if we say that $f(z)$ has n infinities in a primitive region of which p are at infinity, then these last p infinities are exponential infinities of $f(z)$ in a primitive region while the other $n - p$ are poles.

§19. It follows from the new definitions that *a circular function of the first order assumes the values of its characteristic limits only once in a primitive region.* In fact, if ω be the period of such a function, we have by formula (15)

$$f(z) = \frac{mu_{\omega} + n}{pu_{\omega} + q}.$$

Let f_1 and f_2 be the characteristic limits of $f(z)$ in the positive ($u_{\omega} = 0$) and respectively negative ($u_{\omega} = \infty$) direction. Then

$$f_1 = \frac{n}{q}; \quad f_2 = \frac{m}{p}.$$

We will first assume that p and q are not equal to zero. Then

$$\begin{aligned} \lim_{\substack{z=\infty \\ u_{\omega}=\infty}} u_{\omega} [f(z) - f_2] &= \frac{np - mq}{p^2} \begin{cases} \neq 0 \\ \neq \infty \end{cases}, \\ \lim_{\substack{z=\infty \\ u_{\omega}=0}} \left[\frac{f(z) - f_1}{u_{\omega}} \right] &= \frac{mq - np}{q^2} \begin{cases} \neq 0 \\ \neq \infty \end{cases}, \end{aligned}$$

which proves the proposition. If, on the other hand, $q = 0$ or $p = 0$, the function $f(z)$ takes one or the other of the two forms: $C + \frac{C'}{u_{\omega}}$ or $Cu_{\omega} + C'$. In either case the proposition becomes obvious.

If we take into account the remark made in §17, namely, that a circular function of the first order assumes the values of its characteristic limits within a primitive region only at these limits, then the proposition of this paragraph may be enunciated in the form of the following

THEOREM.—*A circular function of the first order assumes every assignable value only once in a primitive region.*

§20. Let $f(z)$ be a circular function admitting the period ω , and let in a primitive region $\alpha_1, \alpha_2, \dots, \alpha_p$ be its vanishing points of the respective orders of multiplicity $\lambda_1, \lambda_2, \dots, \lambda_p$; $\beta_1, \beta_2, \dots, \beta_q$ its poles of the respective orders $\mu_1, \mu_2, \dots, \mu_q$; moreover, let the characteristic limits of $f(z)$ be finite and $\neq 0$. Then (§15) $\sum_1^p \lambda_\kappa = \sum_1^q \mu_\kappa = n$, n being the order of the circular function.

Let $\theta(z)$ be a circular function of the first order admitting the period ω and which is finite and $\neq 0$ in the finite portion of a primitive region. Then the function

$$\Theta(z) = \frac{[\theta(z) - \theta(\alpha_1)]^{\lambda_1} [\theta(z) - \theta(\alpha_2)]^{\lambda_2} \dots [\theta(z) - \theta(\alpha_p)]^{\lambda_p}}{[\theta(z) - \theta(\beta_1)]^{\mu_1} [\theta(z) - \theta(\beta_2)]^{\mu_2} \dots [\theta(z) - \theta(\beta_q)]^{\mu_q}} \quad (19)$$

is a circular function with the period ω , whose vanishing points and poles are the same and of the same respective orders of multiplicity as those of $f(z)$. Moreover, the characteristic limits of $\Theta(z)$ are finite and $\neq 0$,

$$\begin{aligned} \Theta_1 &= 1, \\ \Theta_2 &= \frac{[-\theta(\alpha_1)]^{\lambda_1} [-\theta(\alpha_2)]^{\lambda_2} \dots [-\theta(\alpha_p)]^{\lambda_p}}{[-\theta(\beta_1)]^{\mu_1} [-\theta(\beta_2)]^{\mu_2} \dots [-\theta(\beta_q)]^{\mu_q}}. \end{aligned}$$

Hence (Theorem I, §14)

$$f(z) = C\Theta(z), \quad (19)'$$

i. e. the function $f(z)$ is a rational function of $\theta(z)$. The degree of the numerator in (19) is equal to that of the denominator and to n , *i. e.* to the order of the function $f(z)$. This fact is due to the condition that the characteristic limits of $f(z)$ be finite and $\neq 0$.

Let us now consider any circular function of the n^{th} order. We can always find two numbers A and B such that the characteristic limits of the circular function of the n^{th} order

$$F(z) = A + \frac{1}{f(z) + B}$$

be finite and $\neq 0$ (§16). Then, as has been just proved, $F(z)$ is a rational function of the circular function of the first order $\theta(z)$,

$$F(z) = \frac{a + b\theta(z) + c[\theta(z)]^2 + \dots + g[\theta(z)]^n}{a' + b'\theta(z) + c'[\theta(z)]^2 + \dots + g'[\theta(z)]^n},$$

in which a, a', g, g' are all $\neq 0$, and therefore $f(z)$ is again a rational function of $\theta(z)$, namely,

$$f(z) = \frac{a_1 + b_1\theta(z) + \dots + g_1[\theta(z)]^n}{a_2 + b_2\theta(z) + \dots + g_2[\theta(z)]^n},$$

where at least one of the coefficients a_1 and a_2 and at least one of the coefficients g_1 and g_2 are $\neq 0$, because otherwise the order of the function $f(z)$ would be less than n .

We have seen (§16) that every circular function of the first order can be expressed in form of a linear fractional function of any other circular function of the first order. Hence this

THEOREM.—*Every circular function of the n^{th} order $f(z)$ can be represented as a rational function of any assigned circular function of the first order $\phi(z)$*

$$f(z) = \frac{a + b\phi(z) + \dots + g[\phi(z)]^n}{a' + b'\phi(z) + \dots + g'[\phi(z)]^n}, \quad (20)$$

where at least one of the coefficients a and a' and one of the coefficients g and g' are $\neq 0$.

The coefficients g and g' cannot both vanish, because then $f(z)$ would be a circular function of the order $n - 1$ at the most. For a similar reason a and a' cannot both vanish, because otherwise we could cancel the common factor $\phi(z)$ in the numerator and in the denominator of the expression (20), and then the order of $f(z)$ would be again $n - 1$ at the most.

COROLLARY.—*Circular functions admit an algebraic addition theorem.*

In fact let us put $\phi(z) = u_\omega$ in (20), then

$$f(z) = \frac{P(u_\omega)}{Q(u_\omega)}, \quad (20)'$$

where $P(u_\omega)$ and $Q(u_\omega)$ are polynomials in u_ω . If we denote $\phi(t)$ by v_ω we shall have in a similar way

$$f(t) = \frac{P(v_\omega)}{Q(v_\omega)}, \quad (20)''$$

$$f(z + t) = \frac{P(u_\omega v_\omega)}{Q(u_\omega v_\omega)}. \quad (20)'''$$

Eliminating u_ω and v_ω between the three equations $(20)'$, $(20)''$, $(20)'''$, we obtain an expression of the form

$$F[f(z), f(t), f(z+t)] = 0,$$

where F denotes an algebraic function. Q. E. D.

§21. Let us take again $\phi(z) \equiv u_\omega$ in formula (20). According as $a \neq 0$ or $a' \neq 0$ we have the two forms:

$$f(z) = \frac{a_0 + a_1 u_\omega + a_2 u_\omega^2 + \dots + a_p u_\omega^p}{b_\kappa u_\omega^\kappa + b_{\kappa+1} u_\omega^{\kappa+1} + \dots + b_q u_\omega^q}, \quad (21)$$

$$f(z) = \frac{a_\kappa u_\omega^\kappa + a_{\kappa+1} u_\omega^{\kappa+1} + \dots + a_p u_\omega^p}{b_0 + b_1 u_\omega + b_2 u_\omega^2 + \dots + b_q u_\omega^q}, \quad (22)$$

where at least one of the numbers p or q is equal to n , i. e. to the order of the circular function $f(z)$.

A first result of these expressions is that *if a characteristic limit of a circular function is infinite (or zero), then the function has an exponential infinity (or an exponential zero) of some order at this limit.* In fact, according as $p \geq q$ or $p \leq q$, formulas (21) and (22) show that

$$\begin{aligned} (p \geq q) \quad \lim_{\substack{z=\infty \\ u_\omega=\infty}} \left[\frac{f(z)}{u_\omega^{p-q}} \right] &= \frac{a_p}{b_q} \left\{ \begin{array}{l} \neq 0 \\ \neq \infty \end{array} \right., \\ (q \geq p) \quad \lim_{\substack{z=\infty \\ u_\omega=\infty}} [u_\omega^{q-p} f(z)] &= \frac{a_p}{b_q} \left\{ \begin{array}{l} \neq 0 \\ \neq \infty \end{array} \right., \end{aligned}$$

and according as we have the form (21) or the form (22),

$$\begin{aligned} [\text{form (21)}] \quad \lim_{\substack{z=\infty \\ u_\omega=0}} [u_\omega^\kappa f(z)] &= \frac{a_0}{b_\kappa} \left\{ \begin{array}{l} \neq 0 \\ \neq \infty \end{array} \right., \\ [\text{form (22)}] \quad \lim_{\substack{z=\infty \\ u_\omega=0}} \left[\frac{f(z)}{u_\omega^\kappa} \right] &= \frac{a_\kappa}{b_0} \left\{ \begin{array}{l} \neq 0 \\ \neq \infty \end{array} \right.. \end{aligned}$$

The order of the exponential infinity or of the exponential zero in the negative direction is always $|p-q|$, the order of the exponential infinity or of the exponential zero in the positive direction always κ . If $p > q$, then the characteristic limit in the negative direction is an exponential infinity (of the order $p-q$), and if $p < q$ then it is an exponential zero (of the order $q-p$). When the function $f(z)$ has the form (21) then the characteristic limit in the positive

direction is an exponential infinity (of the order κ), and if $f(z)$ has the form (22) then this characteristic limit is an exponential zero (of the order κ).

If one of the characteristic limits of $f(z)$ is to be finite and $\neq 0$, we must have either $p = q = n$ (if the finite characteristic limit is in the negative direction) or $\kappa = 0$ (if it is to be in the positive direction), and if both characteristic limits are to be finite we must have

$$p = q = n; \quad \kappa = 0,$$

and we thus obtain again the theorem expressed by the equation (19)' (§20).

The vanishing points and poles of $f(z)$ in a primitive region are given respectively by the equations

$$[\text{form (21)}] \quad \begin{cases} a_0 + a_1 u_\omega + \dots + a_p u_\omega^p = 0, \\ b_\kappa + b_{\kappa+1} u_\omega + \dots + b_q u_\omega^{q-\kappa} = 0; \end{cases}$$

$$[\text{form (22)}] \quad \begin{cases} a_\kappa + a_{\kappa+1} u_\omega + \dots + a_p u_\omega^{p-\kappa} = 0, \\ b_0 + b_1 u_\omega + \dots + b_q u_\omega^q = 0. \end{cases}$$

To each root of these equations corresponds one determinate point in a primitive region. Hence we have in the first case p vanishing points and $q - \kappa$ poles; in the second, $p - \kappa$ vanishing points and q poles. Combining these results with those obtained with regard to the exponential infinities and zeros, we shall have, if $p \geq q$ (then $p = n$),

$$[\text{form (21)}] \quad \begin{cases} p \text{ vanishing points.} \\ q - \kappa \text{ poles; } (p - q) + \kappa \text{ exponential infinities.} \end{cases}$$

$$[\text{form (22)}] \quad \begin{cases} p - \kappa \text{ vanishing points; } \kappa \text{ exponential zeros.} \\ q \text{ poles; } p - q \text{ exponential infinities,} \end{cases}$$

and if $p \leq q$ (then $q = n$),

$$[\text{form (21)}] \quad \begin{cases} p \text{ vanishing points; } q - p \text{ exponential zeros.} \\ q - \kappa \text{ poles; } \kappa \text{ exponential infinities.} \end{cases}$$

$$[\text{form (22)}] \quad \begin{cases} p - \kappa \text{ vanishing points; } \kappa + (q - p) \text{ exponential zeros.} \\ q \text{ poles.} \end{cases}$$

This table shows that in all cases *the circular function $f(z)$ of the n^{th} order vanishes as many times as it becomes infinite in a primitive region, namely, n times.*

More generally, we can now enunciate the following

THEOREM.—*A circular function of the n^{th} order assumes any arbitrarily assigned value exactly n times in a primitive region.*

Indeed to prove this theorem we only need to consider the function $\phi(z) = f(z) - N$ where N is any arbitrarily assigned number, and to apply to the function $\phi(z)$ the results previously obtained.

REMARK.—If $f(z)$ is finite and $\neq 0$ in the finite portion of a primitive region, then it has the form

$$f(z) = Cu_{\omega}^{\pm n},$$

as can be readily seen from the formulas (21) and (22).

§22. Let us put in evidence the vanishing points and poles of $f(z)$ in a primitive region. Let $\alpha_1, \alpha_2, \dots, \alpha_p$ be the vanishing points and $\lambda_1, \lambda_2, \dots, \lambda_p$ their respective orders; $\beta_1, \beta_2, \dots, \beta_q$ the poles and $\mu_1, \mu_2, \dots, \mu_q$ their respective orders, in a primitive region of $f(z)$. We will denote the value of u_{ω} at a vanishing point α_{κ} by v_{κ} and at a pole β_{κ} by w_{κ} . Then both forms (21) and (22) of the function $f(z)$ will be contained in the formula

$$f(z) = cu_{\omega}^{\nu} \frac{(u_{\omega} - v_1)^{\lambda_1} (u_{\omega} - v_2)^{\lambda_2} \dots (u_{\omega} - v_p)^{\lambda_p}}{(u_{\omega} - w_1)^{\mu_1} (u_{\omega} - w_2)^{\mu_2} \dots (u_{\omega} - w_q)^{\mu_q}}, \quad (23)$$

where ν may be a positive or negative integer or zero.*

Let us put

$$\rho = \nu + \sum_1^p \lambda_{\kappa} - \sum_1^q \mu_{\kappa}. \quad (24)$$

Then, according as $\rho > 0$ or $\rho < 0$, the characteristic limit of $f(z)$ in the negative direction ($u_{\omega} = \infty$) will be an exponential infinity or an exponential zero, the order of multiplicity being always equal to $|\rho|$. And according as $\nu > 0$ or $\nu < 0$, the characteristic limit of $f(z)$ in the positive direction ($u_{\omega} = 0$) will be an exponential zero or an exponential infinity, the order of multiplicity being always equal to $|\nu|$.

If we differentiate the log of (23) with respect to z , we obtain

$$\frac{f'(z)}{f(z)} = \frac{2\pi i}{\omega} \left\{ \nu + \sum_1^p \frac{\lambda_{\kappa} u_{\omega}}{u_{\omega} - v_{\kappa}} - \sum_1^q \frac{\mu_{\kappa} u_{\omega}}{u_{\omega} - w_{\kappa}} \right\}, \quad (25)$$

* It will be noticed that the function (23) is a circular function when ν is any rational number, but ν must be an integer or zero if the period of $f(z)$ is to be ω . If we had $\nu = \frac{\kappa}{s}$, where κ and s are integers and the fraction is irreducible, then the period of $f(z)$ would be $s\omega$.

and making z tend to infinity in the negative, respectively positive direction

$$\lim_{\substack{z=\infty \\ u_\omega=\infty}} \left[\frac{f'(z)}{f(z)} \right] = \frac{2\pi i}{\omega} \rho; \quad \lim_{\substack{z=\infty \\ u_\omega=0}} \left[\frac{f'(z)}{f(z)} \right] = \frac{2\pi i}{\omega} \nu.$$

A first result from these formulas is that the function $f'(z)$, which is evidently simply periodic like $f(z)$, tends to a determinate limit when z tends to infinity in a fixed direction while remaining in a primitive region. Hence (§10) this

THEOREM I.—*The derivative $f'(z)$ of a circular function $f(z)$ admitting the period ω is also a circular function having the same period.*

We have seen that when a characteristic limit of a circular function is finite and $\neq 0$, then $\rho = 0$ (if it is the characteristic limit in the negative direction) or $\nu = 0$ (if it is the characteristic limit in the positive direction), and that $\rho = \nu = 0$ if both characteristic limits of $f(z)$ are finite and $\neq 0$. Hence, by (26), we have the

THEOREM II.—*When a characteristic limit of a circular function $f(z)$ is finite and $\neq 0$, then the corresponding* characteristic limit of the derivative function $f'(z)$ vanishes, i. e. it is an exponential zero.*

This exponential zero is, in general, of the first order, but it may be of an order higher than the first. In fact formula (25) gives according as f_2 (characteristic limit of $f(z)$ in the negative direction) or f_1 (characteristic limit in the positive direction) is finite and $\neq 0$,

$$f_2 \begin{cases} \neq 0 \\ \neq \infty \\ (\rho=0) \end{cases} \quad \frac{f'(z)}{f(z)} = \frac{2\pi i}{\omega} \left\{ \sum_1^p \frac{\lambda_\kappa v_\kappa}{u_\omega - v_\kappa} - \sum_1^q \frac{\mu_\kappa w_\kappa}{u_\omega - w_\kappa} \right\},$$

$$f_1 \begin{cases} \neq 0 \\ \neq \infty \\ (\nu=0) \end{cases} \quad \frac{f'(z)}{f(z)} = \frac{2\pi i}{\omega} \left\{ \sum_1^p \frac{\lambda_\kappa u_\omega}{u_\omega - v_\kappa} - \sum_1^q \frac{\mu_\kappa u_\omega}{u_\omega - w_\kappa} \right\}.$$

In the first case we have

$$\lim_{\substack{z=\infty \\ u_\omega=\infty}} [f'(z) u_\omega] = \frac{2\pi i}{\omega} f_2 \left\{ \sum_1^p \lambda_\kappa v_\kappa - \sum_1^q \mu_\kappa w_\kappa \right\}.$$

* i. e. in the same direction.

In the second,

$$\lim_{\substack{z=\infty \\ u_\omega=0}} \left[\frac{f'(z)}{u_\omega} \right] = \frac{2\pi i}{\omega} f_1 \left\{ -\sum_1^p \frac{\lambda_\kappa}{v_\kappa} + \sum_1^q \frac{\mu_\kappa}{w_\kappa} \right\}.$$

Hence, provided that $\sum_1^p \lambda_\kappa v_\kappa \neq \sum_1^q \mu_\kappa w_\kappa$ or that $\sum_1^p \frac{\lambda_\kappa}{v_\kappa} \neq \sum_1^q \frac{\mu_\kappa}{w_\kappa}$, the corresponding characteristic limit of $f'(z)$ will be an exponential zero of the first order. But if in the first case $\sum \lambda_\kappa v_\kappa = \sum \mu_\kappa w_\kappa$ or if in the second $\sum \frac{\lambda_\kappa}{v_\kappa} = \sum \frac{\mu_\kappa}{w_\kappa}$, then the corresponding characteristic limit of $f'(z)$ will be an exponential zero of an order higher than the first. That it will in all cases be an exponential zero of a certain order follows from the fact that $f'(z)$ is a circular function (§21).

Suppose now that one of the characteristic limits of $f(z)$ is not finite and $\neq 0$. Then either $\rho \neq 0$ or $\nu \neq 0$. Formulas (26) show that in this case the corresponding characteristic limit of $f'(z)$ is an exponential infinity or an exponential zero, according as the characteristic limit of $f(z)$ is infinite or zero; moreover, the order of multiplicity of the infinity or zero is the same for the function and for its derivative. Hence this

THEOREM III.—*If a characteristic limit of a circular function is an exponential infinity (or an exponential zero) of an order m , then the corresponding characteristic limit of the derivative function is also an exponential infinity (or an exponential zero) of the same order m .*

§23. The last theorem enables us to determine the order of the derivative function $f'(z)$ given the order n of the function $f(z)$ and the number q of its *distinct* poles in a primitive region. In fact, let $\beta_1, \beta_2, \dots, \beta_q$ be the poles of $f(z)$ in a primitive region and $\mu_1, \mu_2, \dots, \mu_q$ their respective orders of multiplicity. Let also μ and μ' be the respective orders of the exponential infinities of $f(z)$ in the positive and in the negative direction.

Each pole β_i of $f(z)$ is a pole of the order $\mu_i + 1$ of the derivative function $f'(z)$. And the characteristic limits of $f'(z)$ are exponential infinities of the orders μ and μ' respectively. The function $f'(z)$ has no other infinities in the same primitive region. Hence it becomes infinite exactly $\sum_1^q \mu_\kappa + \mu + \mu' + q$ times

in a primitive region. But $\sum_1 \mu_\kappa + \mu + \mu' = n$, hence the order of $f'(z)$ is $n + q$. We have thus the

THEOREM.—*If q denote the number of distinct poles of a circular function $f(z)$ of the n^{th} order in a primitive region, then the order of the derivative function $f'(z)$ is $n + q$.*

§24. MÉRAY'S THEOREM.*—*If the two characteristic limits f_1 and f_2 of a circular function $f(z)$ are finite, then the sum of its residues with respect to points in a primitive region is equal to*

$$\frac{\omega}{2\pi i} (f_2 - f_1),$$

where as before f_1 is the characteristic limit in the positive direction ($u_\omega = 0$).

Let us take two points a and b on a line $L_{\kappa-1}$ and their congruents on the line L_κ as on the figure of §15. It is readily seen that the positive direction of the line $L_{\kappa-1}$ is that from b to a . If then we take the points a and b sufficiently far from the origin of the plane we shall have

$$|f(z) - f_1| < \varepsilon \quad (27)$$

for all the points on the line aa' , and

$$|f(z) - f_2| < \varepsilon \quad (28)$$

for those of the line bb' .

Now, we know that the sum of the residues of $f(z)$ with respect to all points within the parallelogram ($abb'a'$) is equal to the integral

$$J = \frac{1}{2\pi i} \int f(z) dz$$

taken along the boundary of this parallelogram in the positive direction, which is obviously *opposed* to the positive direction of the line L . We then have

$$2\pi i J = \int_a^b f(z) dz + \int_b^{b'} f(z) dz + \int_{b'}^{a'} f(z) dz + \int_{a'}^a f(z) dz.$$

*Op. cit. vol. I, p. 274. I have given this theorem in my lectures on the Theory of Functions at the Johns Hopkins University in 1895 before the appearance of the second volume of M. Méray's treatise.

The first and the third of the integrals on the right-hand side of this equation evidently cancel each other. At the same time we have, by (27) and (28),

$$\left| \int_b^{b'} f(z) dz - \omega f_2 \right| < \varepsilon \omega,$$

$$\left| \int_a^{a'} f(z) dz - \omega f_1 \right| < \varepsilon \omega,$$

and therefore if we let the points a and b tend to infinity in the positive, respectively negative direction, we obtain in the limit

$$2\pi i J = \omega (f_2 - f_1). \quad \text{Q. E. D.}$$

§25. THEOREM.—*Given a circular function $f(z)$ of the n^{th} order and with the period ω , let A and B be any two numbers other than the characteristic limits of $f(z)$; if a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be respectively the roots of the equations $f(z) = A$ and $f(z) = B$ in a primitive region, we shall have*

$$\sum_1^n a_\kappa - \sum_1^n b_\kappa \equiv \frac{\omega}{2\pi i} \log \frac{(f_1 - B)(f_2 - A)}{(f_1 - A)(f_2 - B)} \pmod{\omega}. \quad (29)$$

Let us first assume that f_1 and f_2 are finite, and let C be any number other than f_1 and f_2 . Consider the integral

$$\frac{1}{2\pi i} \int \frac{zf'(z)}{f(z) - C} dz$$

taken along the boundary of the same parallelogram as in the preceding paragraph. The path of integration being broken into four parts as before, we see first of all that

$$\begin{aligned} \int_a^b \frac{zf'(z)}{f(z) - C} dz + \int_{b'}^{a'} \frac{zf'(z)}{f(z) - C} dz &= \int_a^b \frac{zf'(z)}{f(z) - C} dz - \int_a^b \frac{(z + \omega)f'(z)}{f(z) - C} dz \\ &= -\omega \int_a^b \frac{f'(z)}{f(z) - C} dz = -\omega \lg \left[\frac{f(b) - C}{f(a) - C} \right] \pm 2\pi\omega i. \end{aligned}$$

As to the remaining two integrals, they vanish separately, when a and b tend to infinity in the positive, respectively negative direction. In fact, the characteristic limits of $f'(z)$ being exponential zeros (§22), it is clear that

$$\lim_{\substack{z=\infty \\ u_\omega=0}} [zf'(z)] = 0; \quad \lim_{\substack{z=\infty \\ u_\omega=\infty}} [zf'(z)] = 0,$$

therefore, provided the points a and b are sufficiently far from the origin, we shall have

$$\left| \frac{zf'(z)}{f(z) - C} \right| < \varepsilon$$

for all points on the lines aa' and bb' , and the two integrals

$$\int_a^{a'} \frac{zf'(z)}{f(z) - C} dz; \quad \int_b^{b'} \frac{zf'(z)}{f(z) - C} dz$$

vanish in the limit. Hence in the limit

$$\frac{1}{2\pi i} \int \frac{zf'(z)}{f(z) - C} dz = \frac{\omega}{2\pi i} \log \left(\frac{f_1 - C}{f_2 - C} \right) \pm \kappa\omega.$$

On the other hand, we know that the integral on the left-hand side of the last equation is equal to

$$-\sum_1^n c_\kappa + \sum_1^n \beta_\kappa,$$

where c_1, c_2, \dots, c_n are the roots of the equation $f(z) = C$ in a primitive region and $\beta_1, \beta_2, \dots, \beta_n$ the poles of $f(z)$ in the same region. Hence

$$-\sum_1^n c_\kappa + \sum_1^n \beta_\kappa \equiv \frac{\omega}{2\pi i} \log \left(\frac{f_1 - C}{f_2 - C} \right) \pmod{\omega}.$$

Suppose now that one or both of the characteristic limits of $f(z)$ may be infinite. Let then A and B be numbers other than f_1 and f_2 , and consider the circular function of the n^{th} order

$$\phi(z) = \frac{1}{f(z) - A},$$

whose characteristic limits are finite. The poles of $\phi(z)$ in a primitive region are the roots of the equation $f(z) = A$ in the same region, i. e. the points a_1, a_2, \dots, a_n ; and the roots of the equation $\phi(z) = \frac{1}{B - A}$ are the roots of the equation $f(z) = B$, i. e. the points b_1, b_2, \dots, b_n . Then by the proposition

just proved

$$\sum_1^n a_\kappa - \sum_1^n b_\kappa \equiv \frac{\omega}{2\pi i} \log \left\{ \frac{\frac{1}{f_1 - A} - \frac{1}{B - A}}{\frac{1}{f_2 - A} - \frac{1}{B - A}} \right\} \pmod{\omega}$$

$$\text{Q. E. D.)} \quad \equiv \frac{\omega}{2\pi i} \log \frac{(f_1 - B)(f_2 - A)}{(f_1 - A)(f_2 - B)} \pmod{\omega}$$

COROLLARY I.—*If the characteristic limits of a circular function are equal to each other, then*

$$\sum_1^n a_\kappa \equiv \sum_1^n b_\kappa \pmod{\omega}.$$

The case when both characteristic limits are infinite is here included.

COROLLARY II.—*If only one of the characteristic limits of $f(z)$ is infinite and the other be denoted by f' , then*

$$\sum_1^n a_\kappa - \sum_1^n b_\kappa \equiv \pm \frac{\omega}{2\pi i} \log \left(\frac{f' - A}{f' - B} \right) \pmod{\omega} \quad (30)$$

the upper or lower sign to be taken according as $\lim_{z=\infty} f(z) = \infty$ in the positive or in the negative direction.

§26. Let us now put $f(z) = \zeta$, where ζ is a variable quantity, and let z_1, z_2, \dots, z_n be the roots of this equation. These roots are functions of ζ . By formula (29)

$$\sum_1^n z_\kappa = \frac{\omega}{2\pi i} \log \left(\frac{f_2 - \zeta}{f_1 - \zeta} \right) + \text{constant},$$

provided $\zeta \neq f_1$ or f_2 . If we differentiate this formula with respect to ζ , we obtain

$$\sum_1^n \frac{dz_\kappa}{d\zeta} = \frac{\omega}{2\pi i} \left\{ \frac{1}{f_1 - \zeta} - \frac{1}{f_2 - \zeta} \right\}.$$

But $\frac{dz_\kappa}{d\zeta}$ being one of the values of $\frac{dz}{d\zeta}$ corresponding to the value of ζ , we have

$$\sum_1^n \frac{dz_\kappa}{d\zeta} = \sum_1^n \frac{1}{f'(z_\kappa)}.$$

Hence this

THEOREM.—Given a circular function of the n^{th} order $f(z)$ with the period ω , let z_1, z_2, \dots, z_n be the points in a primitive region at which the function assumes the same value $\zeta \neq f_1$ or f_2 , then

$$\sum_1^n \frac{1}{f'(z'_\kappa)} = \frac{\omega}{2\pi i} \left\{ \frac{1}{f_1 - \zeta} - \frac{1}{f_2 - \zeta} \right\}. \quad (31)$$

COROLLARY.—If the characteristic limits of the function $f(z)$ are equal to each other or infinite, then

$$\sum_1^n \frac{1}{f'(z'_\kappa)} = 0,$$

z_1, z_2, \dots, z_n being the points in a primitive region at which the function $f(z)$ assumes the same value other than the value of the characteristic limits.

§27. THEOREM.—If two circular functions $f(z)$ and $\phi(z)$ have the same period, then they are connected by an algebraic equation.

In fact, these functions can be presented in the form of rational functions of u_ω , i. e. (§20),

$$f(z) = \frac{P(u_\omega)}{Q(u_\omega)}, \quad (32)$$

$$\phi(z) = \frac{P_1(u_\omega)}{Q_1(u_\omega)}. \quad (33)$$

The elimination of u_ω between the equations (32), (33) gives an irreducible algebraic equation of the form

$$F(f(z), \phi(z)) = 0, \quad (34)$$

which proves the theorem. The degree of this equation is, in general, n in $f(z)$ and m in $\phi(z)$, if m and n are the orders of the circular functions $f(z)$ and $\phi(z)$ respectively.* This follows at once from the process of elimination, but there is

* Let

$$f(z) = \frac{a_0 + a_1 u_\omega + \dots + a_p u_\omega^p}{b_0 + b_1 u_\omega + \dots + b_q u_\omega^q}; \quad \phi(z) = \frac{a'_0 + a'_1 u_\omega + \dots + a'_{p'} u_\omega^{p'}}{b'_0 + b'_1 u_\omega + \dots + b'_{q'} u_\omega^{q'}};$$

where at least one of the numbers p or q is $= m$; and at least one of the numbers p' or q' is $= n$. If, then, we put for the sake of brevity

$$b_\kappa f(z) - a_\kappa = x_\kappa; \quad b'_\kappa \phi(z) - a'_\kappa = y_\kappa,$$

another way of arriving to this result. To each value of $\phi(z)$ correspond within a primitive region n points, and in the whole plane an infinity of points, which may be all classified into n groups, each point of a group being congruent with any other point of the same group. To each of these groups corresponds a single value of $f(z)$. Hence to each value of $\phi(z)$ correspond n values of $f(z)$. Likewise to each value of $f(z)$ correspond m values of $\phi(z)$. In general these n values of $f(z)$, respectively the m values of $\phi(z)$ will be distinct, and therefore the irreducible equation will in general be of the degree n in $f(z)$ and of the degree m in $\phi(z)$. But the degrees of the irreducible equation in $f(z)$ and $\phi(z)$ will be lesser if to several groups of points z always corresponds only one value of $f(z)$ or $\phi(z)$. It can be readily shown that in such cases the degrees of the irreducible equation in $f(z)$ and $\phi(z)$ are $n_1 = \frac{n}{D}$ and $m_1 = \frac{m}{D}$ respectively, where D is a common divisor of the numbers n and m ,* and is equal to the

the result of the elimination of u_ω between the two given equations may be written as follows :

$$\begin{vmatrix} x_0 & x_1 & x_2 & \dots & x_m & 0 & 0 & \dots & 0 \\ 0 & x_0 & x_1 & \dots & x_m & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & x_0 & \dots & \dots & \dots & x_m \\ y_0 & y_1 & \dots & y_{n-1} & y_n & 0 & 0 & \dots & 0 \\ 0 & y_0 & \dots & \dots & y_n & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & y_0 & \dots & \dots & \dots & y_n \end{vmatrix} = 0. \quad (34)'$$

The letters x_0, \dots, x_m enter only in the first n lines of this determinant of the $(m+n)^{\text{th}}$ order ; the letters y_0, \dots, y_n only in the last m lines. Therefore this determinant is of the degree n in the function $f(z)$ and of the degree m in the function $\phi(z)$. The equation (34)' being only another form of equation (34), it is clear that the latter will be of the degrees n and m in $f(z)$ and $\phi(z)$ respectively.

To operate the elimination it will, in general, be necessary to add the following $m+n-2$ equations

$$\begin{aligned} u_\omega^\kappa [Q(u_\omega)f(z) - P(u_\omega)] &= 0, \\ \kappa &\equiv 1, 2, \dots, n-1 \\ u_\omega^\kappa [Q_1(u_\omega)\phi(z) - P_1(u_\omega)] &= 0, \\ \kappa &\equiv 1, 2, \dots, m-1. \end{aligned}$$

to the equations (32) and (33), in which case we obtain equation (34)' as the result of the elimination. But in particular cases the number of subsidiary equations necessary to operate the elimination may be less than $m+n-2$. Then the degree of the equation (34) in $f(z)$ and $\phi(z)$ may be less than n and m respectively.

* See Méray, op. cit. vol. II, p. 359.

number of groups of points z which always correspond to the same value of $f(z)$ or $\phi(z)$.

COROLLARY.—*Every circular function $f(z)$ of the n^{th} order is connected with its derivative function $f'(z)$ by an algebraic equation (§22, Theorem I). The degree of this equation is n in $f'(z)$ and $n + q$ in $f(z)$, q being the number of distinct poles of $f(z)$ in a primitive region (§23).*

§28. Let us for the sake of brevity write w for $f(z)$; w' for $f'(z)$; and W for any polynomial in w . Then we may write the irreducible algebraic equation connecting the circular function $f(z)$ with its derivative as follows:

$$W_0 w'^n + W_1 w'^{n-1} + \dots + W_{n-1} w' + W_n = 0 \quad (35)$$

where at least one of the polynomials W_0, W_1, \dots, W_n is of the degree $n + q$. First of all it is clear that $W_0 = \text{constant}$, because otherwise the equation $W_0 = 0$ would determine points such that at these points $w' = \infty$ while w is finite, which is impossible. We may therefore assume that $W_0 = 1$. Further, provided $w \neq f_1$ or f_2 , we have

$$\frac{W_{n-1}}{W_n} = \frac{\omega}{2\pi i} \left\{ \frac{1}{f_2 - w} - \frac{1}{f_1 - w} \right\}. \quad (36)$$

This follows from the relation

$$\frac{W_{n-1}}{W_n} = - \sum_1^n \frac{1}{w'_\kappa} = - \sum_1^n \frac{1}{f'_\kappa(z_\kappa)}$$

and from formula (31).

§29. Without going into a detailed study of the problem of inversion, the solution of the following particular cases will be given here.

PROBLEM I.—*Find all circular functions of the first order.*

Equation (35) becomes in this case either

$$w' = c(w - \alpha)(w - \beta) \quad (37)$$

when $q = 1$ (the characteristic limits of w are finite); or

$$w' = aw + b \quad (38)$$

when $q = 0$ (one of the characteristic limits of w is infinite).

All circular functions of the first order are therefore the inverse of the functions defined by the integrals

$$z = \int \frac{dw}{c(w-\alpha)(w-\beta)} + c'; \quad z = \int \frac{dw}{aw+b} + c. \quad (39)$$

The *moduli of periodicity* ω of these integrals (which are the periods of the inverse functions) are, as easily verified,

$$\omega = \frac{2\pi i}{c(\alpha-\beta)}; \quad \omega = \frac{2\pi i}{a}$$

respectively. The functions w are then

$$w \equiv f(z) = \frac{\alpha - A\beta e^{\frac{2\pi i}{\omega}z}}{1 - Ae^{\frac{2\pi i}{\omega}z}}, \quad (40)$$

$$w \equiv \phi(z) = Ae^{\frac{2\pi i}{\omega}z} + B \quad (41)$$

respectively. The characteristic limits of the circular function $f(z)$ are

$$f_1 = \alpha; \quad f_2 = \beta,$$

and those of the function $\phi(z)$

$$f_1 = B = -\frac{b}{a}; \quad f_2 = \infty.$$

If we put $\alpha = i; \beta = -i; c=1$ in (37), then $f(z)$ is the inverse of the integral

$$\int \frac{dw}{w^2+1},$$

which may be taken as the definition of the function $\operatorname{tg} z$. In this case formula (40) takes the form

$$w \equiv f(z) = A \operatorname{tg} \left(\frac{\pi}{\omega} z + B \right) + C, \quad (42)$$

and therefore we may say that *the functions e^z and $\operatorname{tg} z$ are the two types of the circular functions of the first order.* All such functions are comprised in one or the other of the forms (41) and (42).

PROBLEM II.—*Find all circular functions of the second order whose characteristic limits are infinite.*

In this case $n = 2$; $q = 0$, and taking into consideration that by formula (36) $W_1 = 0$, equation (35) reduces to

$$w'^2 + W_2 = 0.$$

Hence all circular functions of the second order with infinite characteristic limits are the inverse of the functions defined by the integral

$$\int \frac{dw}{\sqrt{a + bw + cw^2}}. \quad (43)$$

If we put here $b = 0$; $a = 1$; $c = -1$, we may take this for the definition of the function $\sin z$. Then all the inverse of the function defined by the integral (43) will be comprised in the general formula

$$w \equiv f(z) = A \sin \left(\frac{2\pi}{\omega} z + B \right) + C. \quad (44)$$

REMARK.—It will be noticed that the characteristic limits of a circular function are the essentially singular points of the inverse function.

§30. To conclude, we will give some general propositions with regard to circular functions of the second order to show their analogy with elliptic functions of the second order.

THEOREM I.—*Let $f(z)$ be a circular function of the second order whose characteristic limits are equal to each other and to the number N . If γ_1, γ_2 be the points in a primitive region at which $f(z)$ assumes the same value $\Gamma \neq N$, then*

$$f(z) = f(\gamma_1 + \gamma_2 - z). \quad (45)$$

For we have in this case $z_1 + z_2 \equiv \gamma_1 + \gamma_2 \pmod{\omega}$, where z_1, z_2 are the points in a primitive region at which the function assumes the same value $f(z)$ (§25, Corollary I), and therefore $f(z_1) = f(z_2) = f(\gamma_1 + \gamma_2 - z_1)$, where z_1 may be any point.

COROLLARY.—*If we take for the origin of the plane a point $\frac{1}{2}(\gamma_1 + \gamma_2)$, then the function of Theorem I will be an even function, i. e. $f(-z) = f(z)$.*

THEOREM II.—*If $N \neq \infty$, then the function of Theorem I has two distinct poles β_1, β_2 of the first order or one pole β of the second order in a primitive region. In the first case $f'(z)$ is a circular function of the fourth order having two exponential*

zeros of the first order and two vanishing points of the first order in a primitive region, namely, the points $\frac{1}{2}(\beta_1 + \beta_2)$, $\frac{1}{2}(\beta_1 + \beta_2 + \omega)$; in the second case $f'(z)$ is of the third order and has two exponential zeros of the first order and one vanishing point of the first order in a primitive region, namely, the point $\beta + \frac{1}{2}\omega$.

THEOREM III.—If $N = \infty$, then the function of Theorem I has no poles in a primitive region, and $f'(z)$ is also a circular function of the second order.

Let then α_1 and α_2 be the vanishing points of $f(z)$ in a primitive region. The vanishing points of $f'(z)$ in a primitive region will be the points $\frac{1}{2}(\alpha_1 + \alpha_2)$ and $\frac{1}{2}(\alpha_1 + \alpha_2 + \omega)$.

The last two propositions follow from the equation

$$f'(z) = -f'(\gamma_1 + \gamma_2 - z)$$

obtained by differentiation from the equation (45).

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